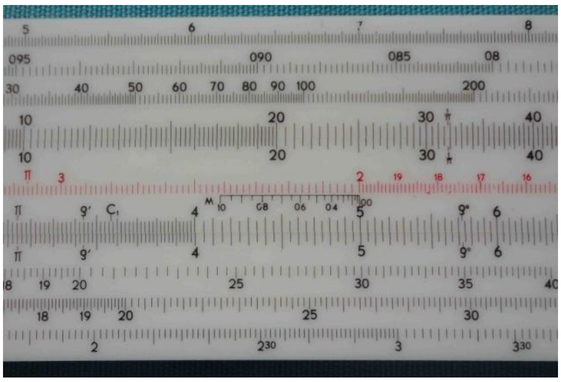
THE “M” SCALE

IN POLISH “SKALA”SLIDE RULES



Angel Carrasco (Roger)

Translated from Spanish by: Alvaro González (Teruteru314) and José Gabriel F[dez]-Raventós (e-lento).

To Martina,
she’s almost here
and we’re awaiting her impatiently.

Madrid, May 2012

ACKNOWLEDGEMENTS

I want to express my sincere gratitude to my colleagues Álvaro González and José Gabriel Fernández, of the Spanish Association “Amigos de las Reglas de Cálculo” (ARC), for their detailed review of the manuscript, providing lots of suggestions and comments, that for sure have enriched the document, and for their kind translation into English.

Also I want to thank Gonzalo Martín, for making his wonderful collection of French rules available to everybody. In particular, I have used the slide rules from Graphoplex brand available in his web www.photocalcul.com, to complete the different models comparison.

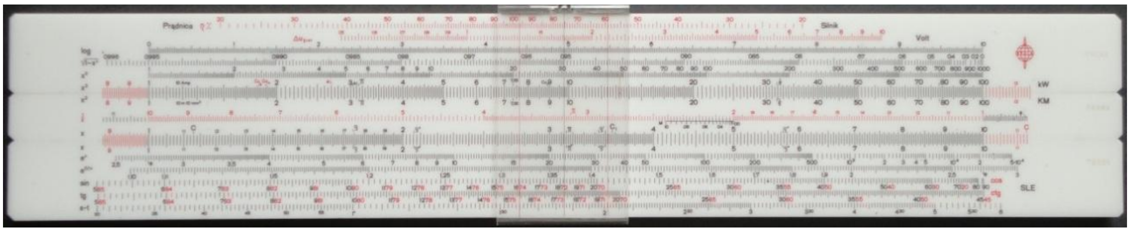
And, of course, I wish to thank Jorge Fábregas, the true responsible for all this, as he created our Association, based in his web www.reglasdecalculo.com.

CONTENT

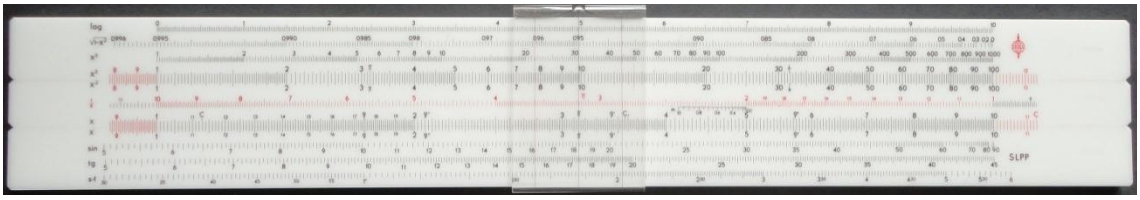
1.- INTRODUCTION

The solution for a right triangle has many practical uses in any technical scope, as the transformation of cartesian into polar coordinates or complex numbers representation in their polar form, defining its modulus and argument from the real and imaginary parts respectively.

The small scale marked with an M letter, appearing in models ELEKTRO SLE and SLPP from polish Brand SKALA, allow easy and fast solutions to right triangles. It additionally gives a better accuracy than the one obtained with traditional trigonometric scales usage.



SKALA ELEKTRO SLE



SKALA SLPP

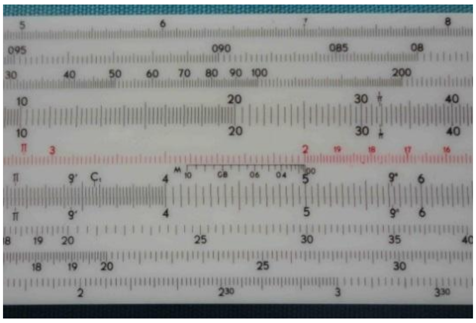
In this article it is described the scale M in detailed form, it’s explained its usage and described the right triangle solution procedures with other types of slide rules through standard trigonometric scales, applying sines theorem.

In the necessarily reduced spaces of slide rules manuals is not possible to explain in detail the theoretical foundations that allow the solution of the different mathematical operations, giving only in its place a series of “recipes”.

We intend to explain those foundations in the belief that it's easier to remember the movements and readings sequences needed on the slide rules scales when the corresponding theoretical base is already known.

2.- SCALE DESCRIPTION

The small scale M sits on the slide, between the scales C and CI, and its length is of about 2 cm.

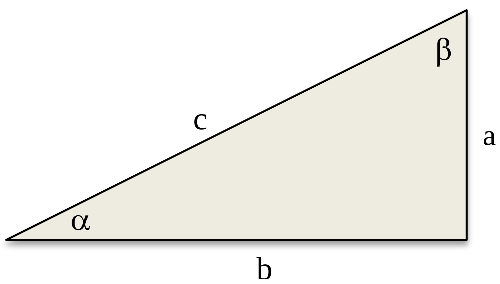


The values of the M scale span from 0 to 1 and increase from right to left. The value 0 of M scale is just over the 5 of C scale, and the value 1 of M is over the 4,14 of C.

3.- SCALE FOUNDATIONS

In a right triangle where its legs are **a** and **b**, the value of the hypotenuse **c** is easily obtained through the direct application of the well known Pythagoras theorem, as:

$$c = \sqrt{a^2 + b^2}$$



That same relationship can be expressed in the following form:

$$c = b + \Delta$$

Being:

$$\Delta = \frac{a^2}{b} \cdot M$$

Where:

$$M = \frac{\sqrt{1+x^2}-1}{x^2}$$

And

$$x = \frac{a}{b}$$

In this quotient, as **a** is the smaller of both legs, it is namely always true:

$$a \leq b$$

The proof that both expressions are really equivalent is very simple, it's enough to exchange the value of **x** in **M** and later the **M** value in **Δ**.

Following the procedure would be:

$$M = \frac{\sqrt{1+\left(\frac{a}{b}\right)^2}-1}{\left(\frac{a}{b}\right)^2}$$

And

$$\Delta = \frac{a^2}{b} \cdot \frac{\sqrt{1+\left(\frac{a}{b}\right)^2}-1}{\left(\frac{a}{b}\right)^2}$$

Operating:

$$\Delta = b \cdot \sqrt{1+\left(\frac{a}{b}\right)^2} - b$$

Substituting this value of Δ, in the first equation becomes:

$$c = b \cdot \sqrt{1+\left(\frac{a}{b}\right)^2}$$

Namely

$$c = b \cdot \sqrt{\frac{a^2+b^2}{b^2}}$$

And finaly

$$c = \sqrt{a^2+b^2}$$

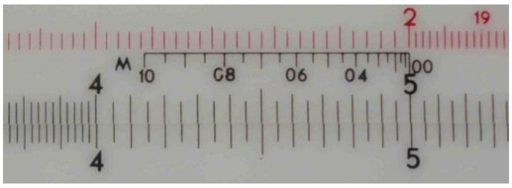
Which demonstrates the equality of the two equations proposed.

By applying this procedure and using the M scale, the value of **Δ** is obtained directly and, by simple addition, the hypotenuse **c** of the triangle.

4.- SCALE PLACEMENT IN THE SLIDE RULE

A relatively interesting aspect which deserves some attention is the position of the scale M in the rule.

First, this is not a conventional full-length scale, but develops from about 4.14 to 5 values of the C scale, having, as already mentioned, a length of 2 cm.



As outlined below, this position is conditioned by the minimum and maximum values that **M** can take as a function of the value of **x**.

The value of x is the ratio between the smallest and largest leg or what is the same, the value of the tangent of the angle between the hypotenuse and the higher leg, which we will call hereinafter **α**, and therefore can only take values between "0" and "1", since the value of the angle **α** shall not exceed 45 degrees.

The value **x = 1** corresponds to the case where **a** and **b** are equal, ie when the triangle, additionally to being right is also isosceles.

In this case it’s easy to see that **M** value is:

$$M = \sqrt{2} - 1 = 0.414$$

What is not so obvious is that when the value of x is infinitely small, practically almost zero, M = 0.5.

To explain this we need to dip into the infinitesimal calculus and the theory of limits.

Indeed, to calculate the value taken by M when x is infinitely small, it is necessary to calculate the limit of the function M, where x has a value close to zero, ie:

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x^2}$$

In reality this situation would never be met and would correspond to the theoretical case in which **a = 0**. In this case it's obvious the triangle can't be constructed, but for very small values of **a** versus **b**, we would obtain values of **x** very close to zero.

If x=0 in the function **M** we obtain what in the theory of limits is known as a type 0/0 indeterminacy.

Such uncertainties in the theory of limits, are solved by applying the L'Hôpital rule.

This rule is named after the 17th century French mathematician Guillaume François Antoine, Marquis de L'Hôpital (1661-1704), who made the rule known in his "*Analyse des infinement petits pour l'intelligence des lignes courbes*" in 1692 .

ANALYSE
DES
INFINIMENT PETITS.
PREMIERE PARTIE.
DU CALCUL DES DIFFERENCES.
SECTION PREMIERE.
Où l'on donne les regles de ce calcul.
DEFINITION I.

The wording of the rule of L'Hopital is essentially as follows:

*If **f(x)** and **g(x)** are two differentiable functions (ie, the derivative function exists) in the neighborhood of a point **p** in which:*

$$f(p) = g(p) = 0$$

*And additionally it is true that the derivative function **g(x)** at point **p** is not null, ie*

$$g'(p) \neq 0$$

Then, if it exists

$$\lim_{x \to p} \frac{f'(x)}{g'(x)}$$

It also exists

$$\lim_{x \to p} \frac{f(x)}{g(x)}$$

And they are the same, ie

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = \lim_{x \to p} \frac{f(x)}{g(x)}$$

Applying the L'Hôpital rule to function M, having in this case p=0, it is:

$$f(x) = \sqrt{1+x^2} - 1$$
$$g(x) = x^2$$

Their derivative functions are:

$$f'(x) = \frac{x}{\sqrt{1+x^2}}$$
$$g'(x) = 2x$$

Then, dividing both numerator and denominator by x, finally it comes to:

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{1}{2\sqrt{1+x^2}} = 0.5$$

In summary, the extreme values of the function **M** are: 0,414 for x = 1 and 0.5 for x = 0, and this explains the position of this scale in the rule, directly confronted with the values between 4.14 and 5.00, respectively, of the C scale.

The exact length of the scale M, for a rule with scales of 250 mm is:

L_M = (log 5.00 – log 4.14)*250 = **20.49 mm**

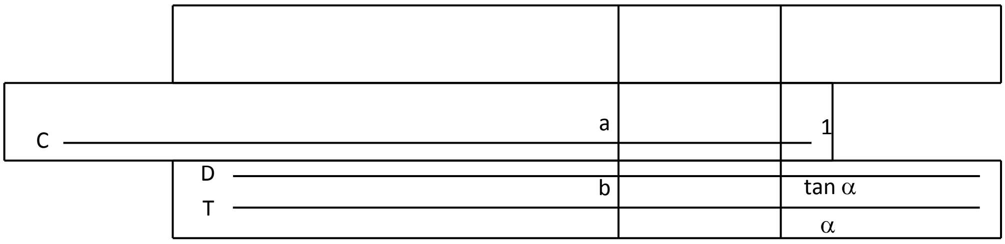
5.- SCALE OPERATION

This section describes in detail the procedure to be followed for the resolution of a right triangle by using the scale **M**.

In the first place and using scales **C** and **D**, calculate the value of **x = tan(α)**, taking into account that **a** corresponds to the smallest of the legs.

This is done easily by placing the value of **a,** on the **C** scale over the value of **b**, on the scale **D**. Under the **1** of the **C** scale, you can read the value of **tan(α)**.

The corresponding angle **α** is obtained directly on the **T** scale, bearing in mind that these rules have angular scales in minutes rather than tenths of degrees.

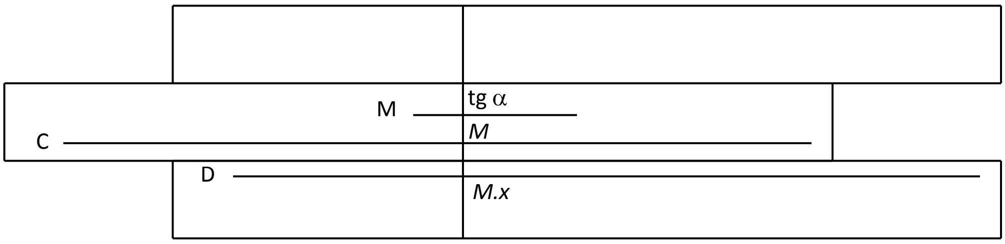


If to obtain the value of **x**, the scale **M** is placed outside the range of the **D** scale, with the help of the cursor the slide should be moved exchanging the left **1** of the C scale for the **10** to the right of the same scale, as in any other calculations with the rule. From here the procedure is the same.

Using the cursor and without moving the slide, move to the value obtained for x on the scale **M**. On the **D** scale will then read the result of the product

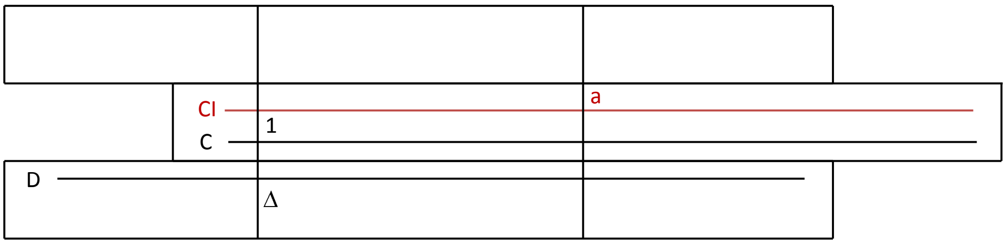
$$M \cdot x = M \cdot \frac{a}{b}$$

Note that in reality the function value of **M** is read on the **C** scale and not on the **M** scale itself.



To finally obtain the value of Δ, is still needed to multiply the above result again by **a**. It is more convenient to use the scale **CI**, ie to divide by the inverse of **a**.

To do this, without moving the cursor, move the slide to read under its main line the value of **a** on the reciprocal scale **CI**. The value of Δ may be read directly on the **D** scale, under the **1** of **C** scale.



Now we just add to this value the one of **b**, to obtain the hypotenuse.

When the value of **x** is very small and therefore difficult to place on the M scale with some precision (since in this area the divisions of the scale are very close together), it can be more efficient to approximate the value of **Δ** with the expression:

$$\Delta = \frac{a^2}{2b} = \frac{a \cdot x}{2}$$

Since, as mentioned above, the value of the **M** function is close to **1/2** where **x** is close to zero.

Furthermore, this latter term is used, in any case, to determine the decimal position on the value of Δ, following the known rules of the estimative calculations, as seen in the following examples.

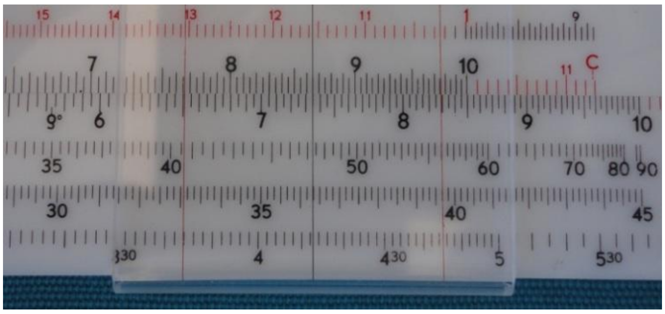
6.- EXAMPLES

Example 1. To solve the triangle whose legs are: a = 7,35 and b = 8,65

Solution:

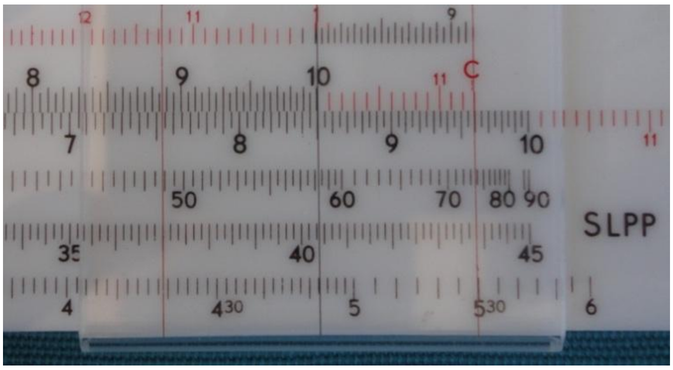
In the first place we calculate the value of **x = tan α**:

$$x = \frac{a}{b} = \frac{7{,}35}{8{,}65} = 0{,}85$$



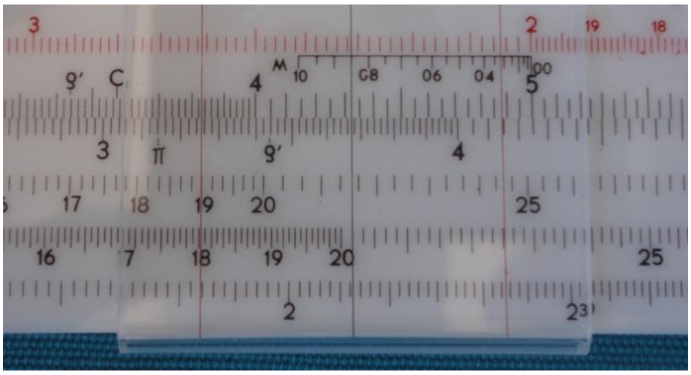
On **T** scale we can read the value of angle **α**, being in this case:

α = 40º 21’



Transferring the result of **x**, on the scale **M**, we can read on the **C** scale, the value of **M**, in this case:

M = 0,432

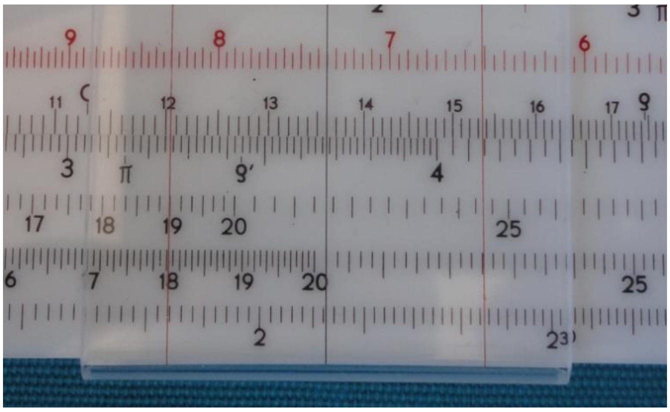


On the **D** scale we can be read, but only has educational interest, the intermediate product:

$$M \cdot x = M \cdot \frac{a}{b}$$

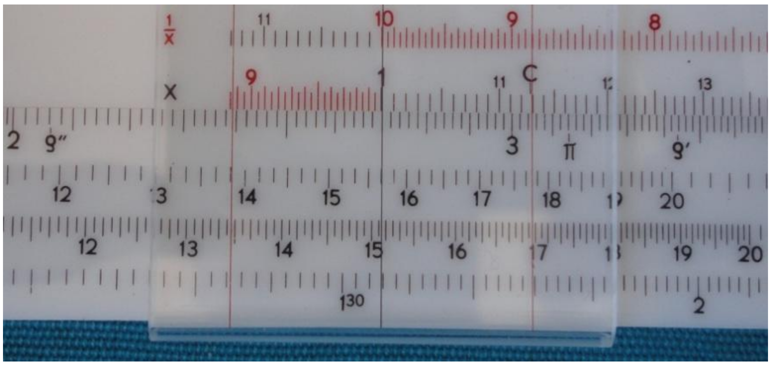
But to finally get the value of **Δ**, we still need to multiply the above result again by **a**.

It is easier, however, to divide by the inverse of **a**, for which, without moving the cursor, **a** value is matched, on the **CI** scale, with the main line of the cursor.



Under the **1** of the **C** scale, the value of **Δ** can be read on **D** scale . In this case:

Δ = 2,7



The decimal position has been obtained by the following aproximation calculation:

$$\frac{a \cdot x}{2} = \frac{7 \cdot 0{,}8}{2} = 2{,}8$$

As seen, it is not necessary to read the intermediate results of operations for obtaining the values of **M** or **M.x**

To get the value of the hypotenuse it is only needed to add the value of **Δ** to the longest leg, that is:

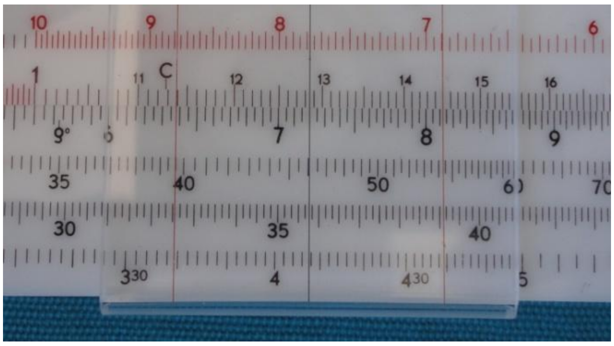
c = Δ + b = 2,70 + 8,65 = **11,35** (exact value: c = 11.351)

Example 2. To solve the triangle whose legs are: a = 7,20 y b = 12,85

Solution:

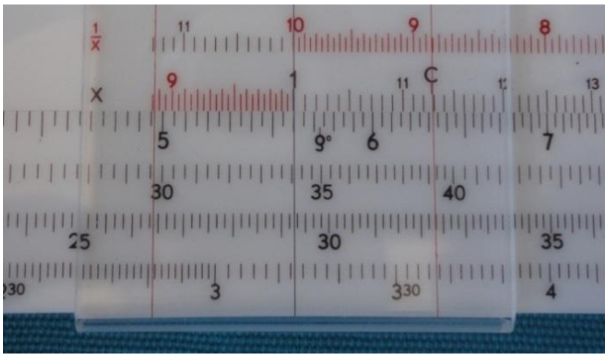
As in the previous case, we calculate the value of **x = tan α**:

$$x = \frac{a}{b} = \frac{7,20}{12,85} = 0,56$$

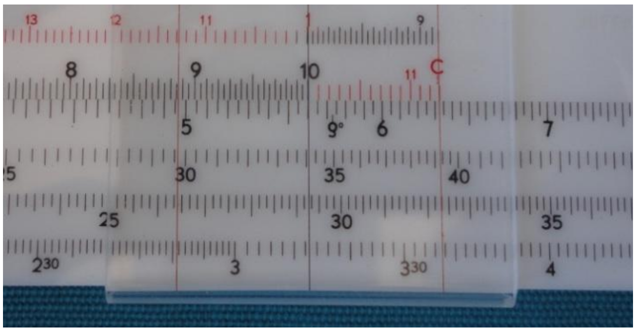


On **T** scale we can read the value of angle **α**, in this case:

α = 29º 15’

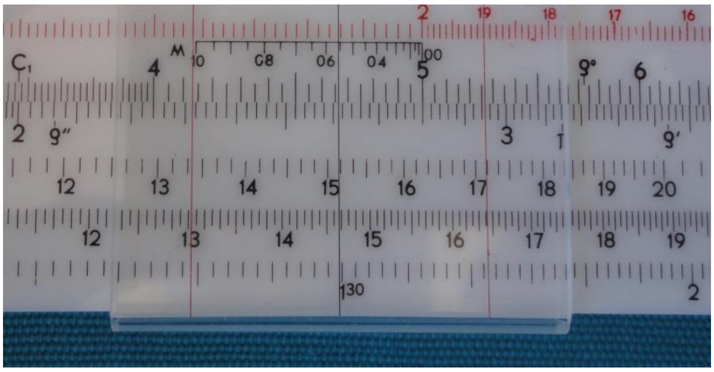


In this case the scale **M** is outside the range of **C** scale and therefore it is necessary to transpose the slide. To do this, without moving the cursor (which is already placed on the **1** of **C** scale), the slide is fully moved to match the **10** of this scale with the main line of the cursor.

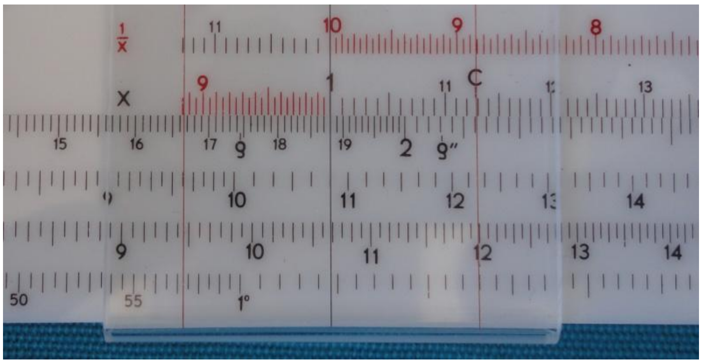


From here the procedure is exactly the same as in the previous example, obtaining the following results:

M = 0,466



Δ = 1,88



Where the decimal position has been obtained by following approximate calculation:

$$\frac{a \cdot x}{2} = \frac{7 \cdot 0{,}5}{2} = 1{,}75$$

And finally:

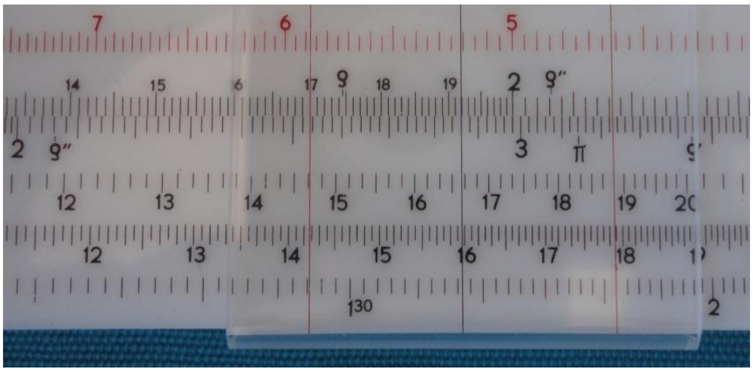
c = Δ + b = 1,88 + 12,85 = **14,73** (exact value c = 14.73)

Example 3. To solve the triangle whose legs are: a = 2,86 y b = 19,20

Solution:

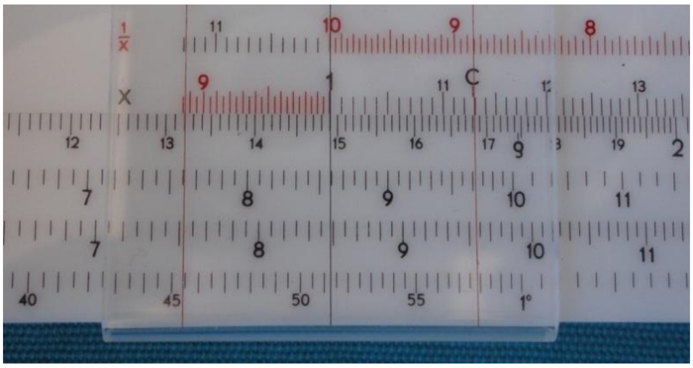
Now the value of **x = tan α** is:

$$x = \frac{a}{b} = \frac{2{,}86}{19{,}20} = 0{,}149$$



On the T scale we can read the value of angle **α**:

α = 8º 28’



Here, where the value of **x** is very small, it is difficult to accurately place it on the scale **M**. Then it is easier and sufficiently approximation to calculate:

$$\Delta = \frac{a^2}{2b} = \frac{a \cdot x}{2} = \frac{2{,}86 \cdot 0{,}149}{2} = 0{,}213$$

In this case the decimals come directly from this last calculation, and it is not necessary any approximate estimation to find the decimal point position.

And finally:

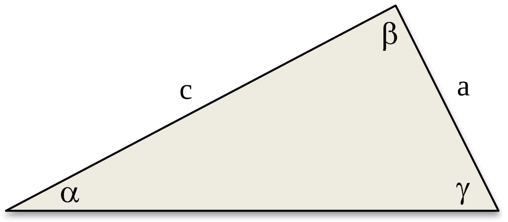
c = Δ + b = 0,213 + 19,20 = **19,413** (exact value c = 19.412)

7.- SOLUTIONS WITH OTHER RULES

The resolution of a triangle with a slide rule can be carried out easily by applying ratios of sines theorem.

$$\frac{a}{sen\alpha} = \frac{b}{sen\beta} = \frac{c}{sen\gamma}$$

Where **a**, **b** and **c** are the sides of the triangle and α, β and γ are the angles opposite each side respectively.



For a right triangle, with the legs **a** and **b,** and **c** the hypotenuse, then:

$$\gamma = 90^{o}$$

And so:

$$sin\ \gamma = 1$$

In addition α and β angles are complementary, ie the two of them add up to 90 º, and thus:

$$sin\ \alpha = cos\ \beta$$
$$cos\ \alpha = sin\ \beta$$

Given the above, the law of sines for a right triangle, can also be written as follows:

$$\frac{a}{sen\alpha} = \frac{b}{\cos\alpha} = c$$

Multiplying all terms by ***sin*** **α:**

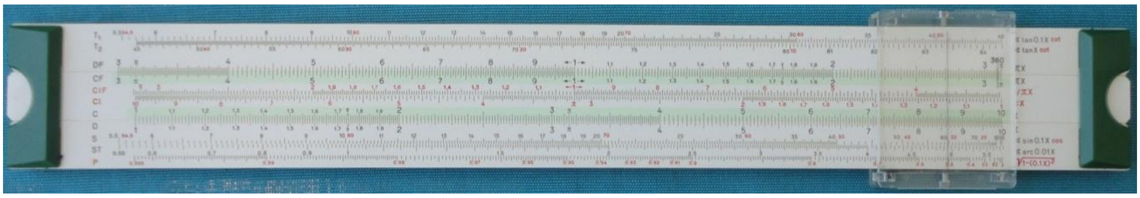
$$a = b \cdot \tan\alpha = c \cdot \sin\alpha$$

Although the theory discussed above is the same, the practical procedure, as discussed below, depends on where the trigonometric scales **S** and **T** are placed in the slide rule. The resolution of triangles, as explained so far, is not possible when the values of the scale **S** are read against the scale of squares **A/B** instead of on the scales **C/D**.

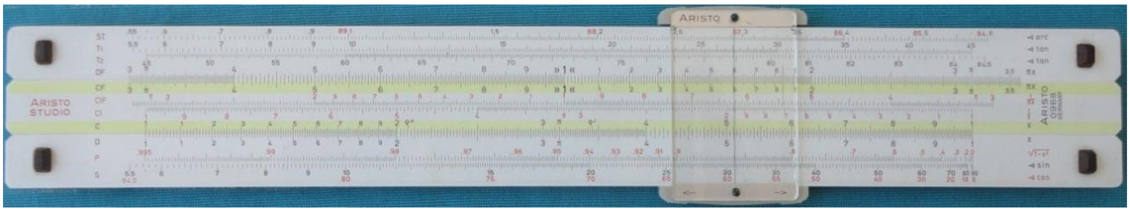
In some cases it will be necessary to transpose the slide, changing with the help of the cursor, the left index (the 1 of C scale) for the right one (10 of the same scale) or vice-versa.

S and T scales on the body of the rule

This arrangement is quite common in European rules, but rules from the French brand Graphoplex and some Aristo models have these scales located on the slide.



FABER-CASTELL 52/82



ARISTO 0968 STUDIO

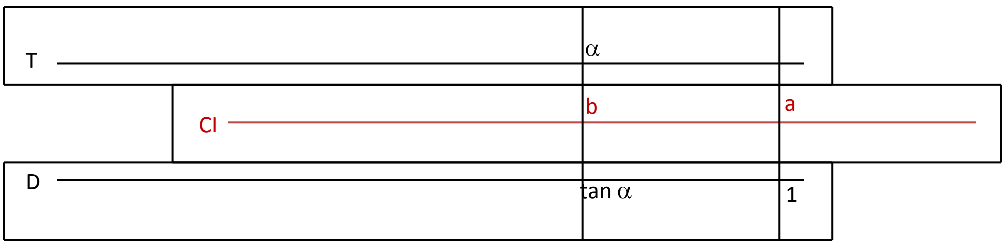
To make use of proportions in this type of slide rule, it is more interesting to express the above equations according to the inverse of **a, b** and **c,** which would finally give:

$$\frac{1}{1/a} = \frac{tg\alpha}{1/b} = \frac{sen\alpha}{1/c}$$

The operating procedure with the slide rule is as follows:

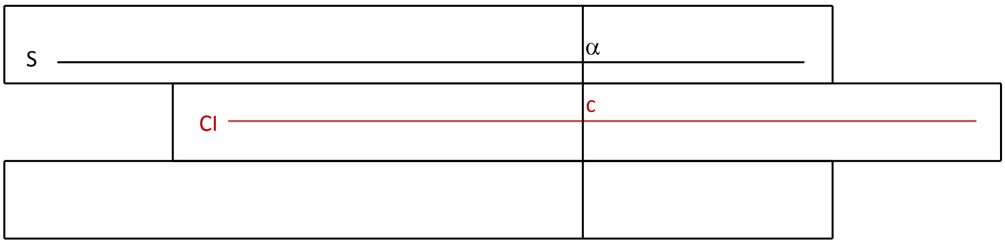
We place the value of **a** (remember that **a** is the shorter leg) on the scale **CI**, above the **10** on the scale **D**.

Move the cursor to the value of **b** on the **CI** scale. On **D** scale we read the value of ***tan α*** and on the **T** scale we can read the value of the angle ***α***.



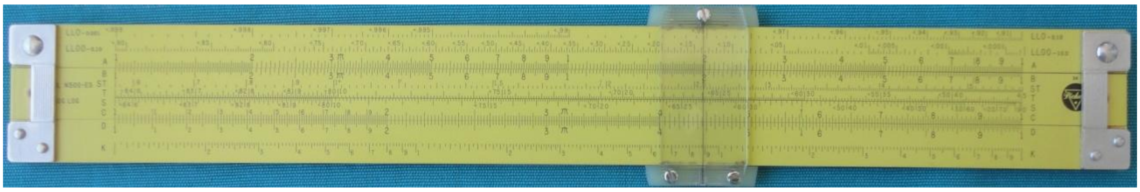
In some cases it is necessary to completely displace the slide to replace the **10** by the **1** on **D** scale.

Without moving the slide, move the cursor to read the same angle **α** on the **S** scale, allowing to read on the **CI** scale the value of the hypotenuse **c**.



S and T scales on the slide

This arrangement is quite common in American-made rules and in the French brand Graphoplex.



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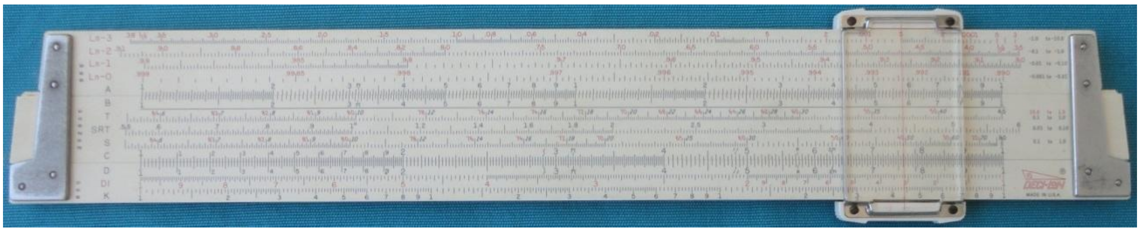

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In this case it is more convenient to remember that the sines law for a right triangle can be expressed also in its inverse form as:

$$\frac{\sin\alpha}{a} = \frac{\cos\alpha}{b} = \frac{1}{c}$$

From this, the only equation with a practical value for the resolution with a slide rule is:

$$\frac{\sin\alpha}{a} = \frac{1}{c}$$

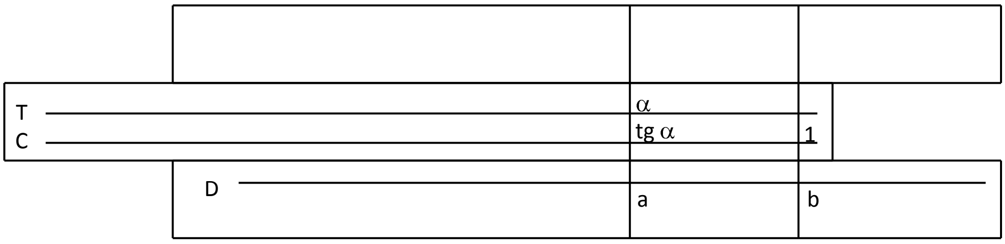
Furthermore, dividing this expression by ***cos α***, it gives

$$\frac{\tan\alpha}{a} = \frac{1}{b} = \frac{1}{c \cdot \cos\alpha}$$

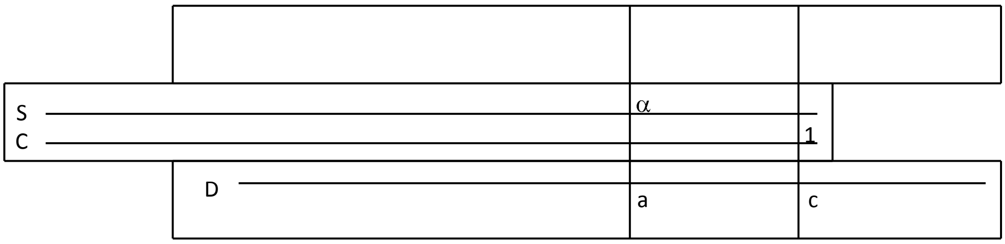
And similarly, here only the first equality is of interest:

$$\frac{tg\,\alpha}{a} = \frac{1}{b}$$

To solve the triangle by such expressions we place the **10** of the **C** scale over the value of **b** (b is the value of the longer leg) on **D** scale. Over the value of **a** in the **D** scale, it can be read the value of ***tan α*** on the **C** scale, and on **T** scale (which is now on the slide) the value of the angle *α*.



Without moving the cursor, move the slide to place the value of **α** from **S** scale below the center line thereof. Then under the **1** of the **C** scale, we can read the value of the hypotenuse **c** in **D** scale. In some cases it is necessary to completely displace the slide to replace the 10 by the 1 of C scale.



S and T scales on the back of the slide

This set of scales is common on "simplex" rules, in which the only scales on the back are the trigonometric scales arranged in the slide.

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In these cases it is more practical to write the above relations as follows:

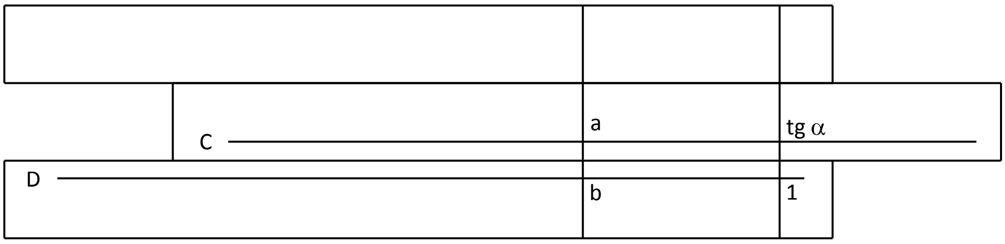
$$\frac{b}{a} = \frac{1}{\tan \alpha}$$

And

$$\frac{c}{a} = \frac{1}{\sin \alpha}$$

With the equations presented in this way, the operation procedure with the rule would be as follows:

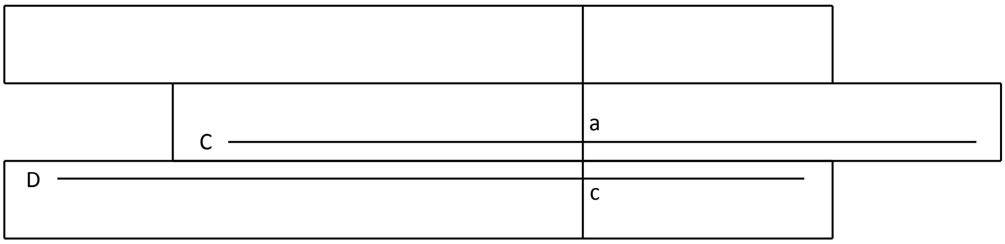
We place the value of the smaller leg **a**, from the **C** scale, over the value of the longer leg **b**, from **D** scale. Then, over the **1** of the **D** scale, we can read the value of ***tan α***



In this position we can be read on the back of the rule, on the **T** scale of the slide, the value of angle **α**.

We move the slide to read the value of **α** on the **S** scale.

Keeping this position of the slide and turning around the rule, on the front and below the value of **a** from **C** scale, you can read the value of the hypotenuse **c**, on the **D** scale.



8.- CONCLUSIONS

In order to stablish a criteria so that to compare the different scale layouts, a summary table is included with the number of movements of the cursor and the slide needed to solve each one of the examples shown.

This table is to evaluate the efficiency of each of the different scale layouts presented.

For counting the movements, the folded scales have not been considered, as these are not available in all slide rules. This type of scales, in some cases, makes the change from left to right end of the scale unnecessary, reducing the number of movements.

In those rules with scales in the back of the rule, models with reading windows at both ends have been used for the counting of movements, so that the reading is available either at the right or the left. On the contrary, some other change from left to right scale ends might be needed, adding to the total number of movements required.

		MOVEMENTS		
		CURSOR	SLIDE	TOTAL
EXAMPLE 1 a = 7.35 b =8.65	SKALA	4	2	6
	S, T in the body	4	2	6
	S, T in the slide front	3	2	5
	S, T in the slide back	4	3	7
EXAMPLE 2 a = 7.20 b = 12.85	SKALA	4	3	7
	S, T in the body	4	2	6
	S, T in the slide front	3	2	5
	S, T in the slide back	3	2	5
EXAMPLE 3 a = 2.86 b = 19.20	SKALA	3	2	5
	S, T in the body	4	2	6
	S, T in the slide front	3	2	5
	S, T in the slide back	3	2	5

As can be appreciated in the preceding table, there are no big differences in the number of movements from one scale layout to the other. The slight differences are due to the need to change of slide ends in some cases.

This difference depends on the problem initial data rather than on the specific scale layout. No scale layout has been found where the change of slide ends had not been needed at some time.

Regarding the precision of the result, as in all slide rules, it is basically dependant on the length of the scales and not on their layout. The bigger the length, the greater is the distance between scale divisions and thus the respective readings are easier.

All models used have 25 cm length scales (except for M scale, obviously), and so this does not bring any difference in regards to results precision.

In relation with the calculation of α angle, the procedure is the same no matter the scale layout of the slide rule, that is, first the value of tg α is found and, afterwards, the value of the angle is found by means of T scale.

Therefore, the precision in the calculation of α angle is exactly the same in all slide rules and is not dependant on the scale layout.

The substantial difference appears in the calculation of the hypotenuse.

In the rules without M scale, independently from the scale layout, the next step is to transfer the value of α to the S scale. The precision of the result depends on how this adjustment is done.

It is easy to see, in this type of slide rules, that the precision increases with the decrease of the value of the angle α, as the divisions in the S scale are larger for small angles. For values of α over 25º (again, it is convenient to remember that α is never greater than 45º) it is not easy to position the value of the angle in the S scale with enough precision. If the angle has decimals of degree, as it is usual, the difficulty is greatly increased.

When the M scale is used, the value to use is not α but tg α, to be placed on M scale. The precision achieved with the use of this scale is much greater, especially for big angle values. For values of a smaller than 10º the precision obtained is similar both with the traditional trigonometric scales and with the M scale.

Also, as it has been shown in example number 3, even in the cases of very low tg α values, where it is somehow difficult to precisely adjust the M scale, the simplified method presented provides values very proximate to the exact value.

To sum up, the rectangle triangle solving with SKALA slide rules where the M scale is included, the value obtained for the α angle has the same precision as when obtained with any other rule, but the difference in the precision for the calculation of the hypotenuse is bigger as bigger is the value of α.

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